Bulk asymptotics of skew-orthogonal polynomials for quartic double well potential and universality in the matrix model

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We derive bulk asymptotics of skew-orthogonal polynomials (sop) $\pi_m^{(\beta)}$, $\beta = 1, 4$, defined w.r.t. the weight $\exp(-2NV(x))$, $V(x) = gx^4/4 + tx^2/2$, $g > 0$ and $t < 0$. We assume that as $m, N \to \infty$ there exists an $\epsilon > 0$, such that $\epsilon \leq (m/N) \leq \lambda_{\rm cr} - \epsilon$, where $\lambda_{\rm cr}$ is the critical value which separates sop with two cuts from those with one cut. Simultaneously we derive asymptotics for the recursive coefficients of skew-orthogonal polynomials. The proof is based on obtaining a finite term recursion relation between sop and orthogonal polynomials (op) and using asymptotic results of op derived in [1]. Finally, we apply these asymptotic results of sop and their recursion coefficients in the generalized Christoffel-Darboux formula (GCD) [8] to obtain level densities and sine-kernels in the bulk of the spectrum for orthogonal and symplectic ensembles of random matrices.

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1. INTRODUCTION

Skew-orthogonal polynomials are useful in the study of orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) ensembles of random matrices [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. In this paper, we derive asymptotics of skew orthogonal functions $\phi_m^{(\beta)}(x)$ and $\psi_m^{(\beta)}(x)$ and their recursion coefficients [8, 9], defined w.r.t. the weight

$$w(x) = \exp(-2NV(x)), \qquad V(x) = \frac{gx^4}{4} + \frac{tx^2}{2}, \qquad g > 0, \qquad t < 0. \tag{1.1}$$

Here, $2N$ is a large parameter which, in the context of random matrix theory, is the size of the matrices.

We define skew-orthogonal functions:

$$\phi_n^{(\beta)}(x) = \frac{1}{\sqrt{g_n^{(\beta)}}}\pi_n^{(\beta)}(x)\exp(-NV(x)), \qquad \pi_n^{(\beta)}(x) = \sum_{k=0}^{n} c_k^{(n,\beta)} x^k, \qquad \beta = 1, 4, \tag{1.2}$$

$$\psi_n^{(4)}(x) := \phi_n'^{(4)}(x), \qquad \psi_n^{(1)}(x) := \int_{\mathbb{R}} \phi_n^{(1)}(y)\epsilon(x-y)dy, \qquad \epsilon(r) = \frac{|r|}{2r}, \qquad n \in \mathbb{N}, \tag{1.3}$$

where $g_n^{(\beta)}$ are normalization constants. They satisfy skew-orthonormality relations:

$$\int_{\mathbb{R}} \phi_n^{(1)}(x)\psi_m^{(1)}(x)dx = Z_{n,m}, \qquad \int_{\mathbb{R}} \phi_n^{(4)}(x)\psi_m^{(4)}(x)dx = \frac{Z_{n,m}}{2}, \qquad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{+} \ldots, \qquad n, m \in \mathbb{N}. \tag{1.4}$$

Using these polynomials, we study the corresponding random matrix model:

$$P_{\beta,N}(H)dH = \frac{1}{\mathcal{Z}_{\beta N}}\exp[-[2{\rm Tr}V(H)]]dH, \qquad \beta = 1, 4, \tag{1.5}$$

where the matrix function $V(H)$ is a double well quartic polynomial of H and

$$\mathcal{Z}_{\beta N} := \int_{H \in M_{2N}^{(\beta)}} \exp[-[2{\rm Tr}u(H)]]dH = (2N)! \prod_{j=0}^{2N-1} g_j^{(\beta)}. \tag{1.6}$$

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Here, $M_{2N}^{(\beta)}$ is a set of all $2N \times 2N$ real symmetric ($\beta = 1$) and quaternion real self dual ($\beta = 4$) matrices. dH is the standard Haar measure.

To study statistical properties of such matrix models, we need to study certain kernel functions [5]:

$$S_{2N}^{(\beta)}(x, y) := \sum_{j,k=0}^{2N-1} Z_{j,k}\phi_j^{(\beta)}(x)\psi_k^{(\beta)}(y), \qquad S_{2N}^{(\beta)}(y, x) = S^{\dagger}{}_{2N}^{(\beta)}(x, y).$$

For quartic potential, $\beta = 1$, this is given by [8]

$$\begin{aligned}
(x-y)S_{2N}^{(1)}(x, y) &= R_{2N-4,2N}^{(1)}[\psi_{2N-3}^{(1)}(x)\psi_{2N}^{(1)}(y) - (x \leftrightarrow y)] + R_{2N-2,2N+2}^{(1)}[\psi_{2N-1}^{(1)}(x)\psi_{2N+2}^{(1)}(y) - (x \leftrightarrow y)] \\
&- R_{2N-3,2N+1}^{(1)}[\psi_{2N-4}^{(1)}(x)\psi_{2N+1}^{(1)}(y) - (x \leftrightarrow y)] - R_{2N-1,2N+3}^{(1)}[\psi_{2N-2}^{(1)}(x)\psi_{2N+3}^{(1)}(y) - (x \leftrightarrow y)] \\
&+ R_{2N-2,2N}^{(1)}[\psi_{2N-1}^{(1)}(x)\psi_{2N}^{(1)}(y) - (x \leftrightarrow y)] - yP_{2N-3,2N}^{(1)}[\psi_{2N-4}^{(1)}(y)\psi_{2N}^{(1)}(x) - (x \leftrightarrow y)] \\
&- [R_{2N-1,2N+1}^{(1)}[\psi_{2N-2}^{(1)}(x)\psi_{2N+1}^{(1)}(y) - (x \leftrightarrow y)] - yP_{2N-1,2N+2}^{(1)}[\psi_{2N-2}^{(1)}(x)\psi_{2N+2}^{(1)}(y) - (x \leftrightarrow y)]] \\
&- yP_{2N-2,2N+1}^{(1)}[\psi_{2N-1}^{(1)}(x)\psi_{2N+1}^{(1)}(y) - (x \leftrightarrow y)] + yP_{2N-1,2N}^{(1)}[\psi_{2N-2}^{(1)}(x)\psi_{2N}^{(1)}(y) - (x \leftrightarrow y)] \qquad (1.7)
\end{aligned}$$

where the recursion coefficients $P_{j,k}^{(1)}$ and $R_{j,k}^{(1)}$ are defined as:

$$\phi_j^{(1)}(x) := \sum_k P_{j,k}^{(1)}\psi_k^{(1)}(x); \qquad x\phi_j^{(1)}(x) := \sum_k R_{j,k}^{(1)}\psi_k^{(1)}(x). \tag{1.8}$$

For $\beta = 4$, $\psi_j^{(1)}(x)$ and $\phi_j^{(1)}(x)$ are replaced by $\phi_j^{(4)}(x)$ and $\psi_j^{(4)}(x)$ respectively [8].

To study asymptotic behavior of these skew-orthogonal functions and their recursion coefficients $P_{j,k}^{(\beta)}$ and $R_{j,k}^{(\beta)}$, we expand $\phi_m^{(4)}(x)$ and $\psi_m^{(1)}(x)$, $m \geq 1$ in a suitable basis such that their derivatives exist and $\phi_m^{(\beta)}(x)$ are polynomials of order m. We obtain recursion relations between sop and the corresponding op. Solving the recursion relations and using asymptotic properties of op [1], we derive asymptotics of sop. Finally, we apply them in the GCD formula (1.7) to study the corresponding matrix models.

A. SOP and Orthogonal Ensemble

For sop with quartic weight, we expand $\psi_m^{(1)}(x) = \gamma_m^{(m)}\phi_{m-3}^{(2)}(x) + \sum^{m-1}\gamma_k^{(m)}\psi_k^{(1)}(x)$ such that $[\psi_m^{(1)}]'(x) = \phi_m^{(1)}(x)$ is a polynomial of order m. Using skew-orthonormality (1.4), we get recursion relations:

$$\begin{aligned}
\psi_{2m+1}^{(1)}(x) &= \gamma_{2m+1}^{(2m+1)}\phi_{2m-2}^{(2)}(x) + \gamma_{2m-1}^{(2m+1)}\psi_{2m-1}^{(1)}(x), \qquad m > 1, &(1.9)\\
\psi_{2m}^{(1)}(x) &= \gamma_{2m}^{(2m)}\phi_{2m-3}^{(2)}(x) + \gamma_{2m-2}^{(2m)}\psi_{2m-2}^{(1)}(x) + \gamma_{2m-4}^{(2m)}\psi_{2m-4}^{(1)}(x), \qquad m > 1, &(1.10)\\
\phi_{2m+1}^{(1)}(x) &= \gamma_{2m+1}^{(2m+1)}\left[P_{2m-2,2m+1}^{(2)}\phi_{2m+1}^{(2)}(x) + \ldots P_{2m-2,2m-5}^{(2)}\phi_{2m-5}^{(2)}(x)\right] + \gamma_{2m-1}^{(2m+1)}\phi_{2m-1}^{(1)}(x), &(1.11)\\
\phi_{2m}^{(1)}(x) &= \gamma_{2m}^{(2m)}\left[P_{2m-3,2m}^{(2)}\phi_{2m}^{(2)}(x) + \ldots P_{2m-3,2m-6}^{(2)}\phi_{2m-6}^{(2)}(x)\right] + \gamma_{2m-2}^{(2m)}\phi_{2m-2}^{(1)}(x) + \gamma_{2m-4}^{(2m)}\phi_{2m-4}^{(1)}(x), &(1.12)
\end{aligned}$$

Here

$$\phi_j^{(2)}(x) = \frac{(x^j + \ldots)}{\sqrt{h_j}}\exp(-NV(x)), \qquad \int \phi_j^{(2)}(x)\phi_k^{(2)}(x)dx = \delta_{j,k}. \tag{1.13}$$

are orthogonal polynomials. For even weight, with $R_m = (h_m/h_{m-1})$, $R_0 = 0$, op satisfy [1]:

$$x\phi_j^{(2)}(x) = \sqrt{R_{j+1}}\phi_{j+1}^{(2)}(x) + \sqrt{R_j}\phi_{j-1}^{(2)}(x); \qquad \frac{d}{dx}\phi_j^{(2)}(x) = \sum_{k=j-3}^{j+3} P_{j,k}^{(2)}\phi_k^{(2)}(x) \tag{1.14}$$

$$P_{j,j+3}^{(2)} = -Ng\sqrt{R_{j+1}R_{j+2}R_{j+3}}, \qquad P_{j,j+1}^{(2)} = -\frac{(j+1)}{2R_{j+1}^{1/2}}, \qquad P_{j,j-1}^{(2)} = \frac{j}{2R_j^{1/2}}, \qquad P_{j,j-3}^{(2)} = Ng\sqrt{R_{j-2}R_{j-1}R_j}.$$

Furthermore, using $\phi^{(2)}_{2m-1}(0) = 0$ in (1.14), we get:

$$\phi^{(2)}_{2m}(0) = (-1)^m \sqrt{\frac{R_{2m-1}\dots R_1}{R_{2m}\dots R_2}}\phi^{(2)}_0(0). \tag{1.15}$$

Using skew-orthogonality relations $((\phi^{(1)}_{2m-2}(x), \psi^{(1)}_{2m+1}(x)) = 0)$, $((\phi^{(1)}_{2m-3}(x), \psi^{(1)}_{2m}(x)) = 0)$ and $((\phi^{(1)}_{2m-1}(x), \psi^{(1)}_{2m}(x)) = 0)$ in (1.9) - (1.12), we get:

$$\begin{aligned} \gamma^{(2m+1)}_{2m-1} &= \gamma^{(2m+1)}_{2m+1}\gamma^{(2m-2)}_{2m-2}P^{(2)}_{2m-5,2m-2}; \qquad \gamma^{(2m)}_{2m-4} = -\gamma^{(2m)}_{2m}\gamma^{(2m-3)}_{2m-3}P^{(2)}_{2m-6,2m-3} \\ \gamma^{(2m)}_{2m-2} &= \gamma^{(2m)}_{2m}\left[\gamma^{(2m-1)}_{2m-1}P^{(2)}_{2m-4,2m-3} + \gamma^{(2m-1)}_{2m-3}\gamma^{(2m-3)}_{2m-3}P^{(2)}_{2m-6,2m-3}\right] \end{aligned} \tag{1.16}$$

respectively. We choose $\gamma^{(2m+1)}_{2m+1} = 1$ and $\gamma^{(2m+1)}_{2m-1} = -\sqrt{R_{2m-2}/R_{2m-3}}$. $\gamma^{(2m-2)}_{2m-2}$, $\gamma^{(2m)}_{2m-2}$ and $\gamma^{(2m)}_{2m-4}$ can be calculated from (1.14) and (1.16). Here, we note that choice of $\gamma^{(2m+1)}_{2m+1}$ and $\gamma^{(2m+1)}_{2m-1}$ is such that we can use properties of orthogonal polynomials to obtain $\psi^{(1)}_{2m+1}(x)$ from (1.9). However, this makes the sop non-monic.

Using (1.14) and (1.16), we solve (1.9). $\psi^{(1)}_{2m+1}(x)$ while $\phi^{(1)}_{2m}(x)$ is obtained using skew-orthonormalization (1.4):

$$\psi^{(1)}_{2m+1}(x) = \sqrt{R_{2m-1}}\left[\frac{\phi^{(2)}_{2m-1}(x)}{x}\right], \qquad \phi^{(1)}_{2m}(x) = \frac{x\phi^{(2)}_{2m-1}(x)}{\sqrt{R_{2m-1}}}, \qquad m > 1. \tag{1.17}$$

$\phi^{(1)}_{2m+1}(x)$ and $\psi^{(1)}_{2m}(x)$ can be obtained by taking derivative of $\psi^{(1)}_{2m+1}(x)$ and using skew-orthonormality respectively.

Now, we derive asymptotics for sop using results for op [1]. In the limit $m, N \to \infty$, we assume $\epsilon > 0$, such that $\epsilon < \lambda(= m/N) < \lambda_{cr}(= t^2/4g)$. This ensures that $\phi^{(\beta)}_m(x)$ and $\psi^{(\beta)}_m(x)$ are concentrated on two intervals $[-x_2, -x_1]$ and $[x_1, x_2]$ and exponentially small outside. Using results from [1] for large m,

$$\phi^{(2)}_m(x) = \frac{2C_m\sqrt{x}}{\sqrt{\sin\theta}}[\cos(f_m(\theta)) + O(N^{-1})], \qquad f_m(\theta) = \left(\frac{m+1/2}{2}\right)\left(\frac{\sin(2\theta)}{2} - \theta\right) - (-1)^m\frac{\chi}{4} + \frac{\pi}{4}, \tag{1.18}$$

$$R_{2m+1}, R_{2m} = R, L + O(N^{-2}) = \frac{-t \pm \sqrt{t^2 - 4\lambda g}}{2g}, \qquad \text{with} \quad \lambda = gRL, \qquad t + g(R+L) = 0. \tag{1.19}$$

Thus we have for $x_{1,2} = \sqrt{(-t \mp 2\sqrt{\lambda g})/g}$, in the range $x_1 + \delta < x < x_2 - \delta$, $\delta > 0$ and $m > 1$,

$$\psi^{(1)}_{2m+1}(x) \simeq 2C_{2m-1}\sqrt{\frac{R}{x\sin\theta}}\left[\cos\left(f_{2m-1}(\theta)\right) + O\left(\frac{1}{N}\right)\right], \qquad \psi^{(1)}_{2m}(x) \simeq \frac{C_{2m-1}}{N\lambda}\sqrt{\frac{Lx}{\sin^3\theta}}\left[\sin\left(f_{2m-1}(\theta)\right) + O\left(\frac{1}{N}\right)\right],$$

$$\phi^{(1)}_{2m+1}(x) \simeq -4N\lambda C_{2m-1}\sqrt{\frac{x\sin\theta}{L}}\left[\sin\left(f_{2m-1}(\theta)\right) + O\left(\frac{1}{N}\right)\right], \phi^{(1)}_{2m}(x) \simeq 2C_{2m-1}\sqrt{\frac{x^3}{R\sin\theta}}\left[\cos\left(f_{2m-1}(\theta)\right) + O\left(\frac{1}{N}\right)\right],$$

where

$$q = \cos\theta = \frac{gx^2 + t}{2\sqrt{\lambda' g}}, \qquad \cos\chi = \frac{2\sqrt{\lambda' g} - tq}{2\sqrt{\lambda' g}q - t}, \qquad \lambda = \frac{m}{N}, \qquad C_m = \frac{1}{2\sqrt{\pi}}\left(\frac{g}{\lambda}\right)^{1/4}(1 + O(N^{-1})). \tag{1.20}$$

Here, we note, that for a given x, θ varies with m. We have chosen $\theta \equiv \theta_{2m-1}$ such that

$$\psi^{(1)}_{2m+1\pm k}(x) \simeq 2C_{2m-1}\sqrt{\frac{R}{x\sin\theta}}\left[\cos\left(f_{2m-1}(\theta) \mp (k\theta)/2\right) + O(1/N)\right], \qquad m >> k, \tag{1.21}$$

and so on for $\psi^{(1)}_{2m\pm k}(x)$, $\phi^{(1)}_{2m+1\pm k}(x)$, $\phi^{(1)}_{2m\pm k}(x)$.

For the recursion coefficients (1.8), we simply read off from (1.9) - (1.12). We get for large m:

$$\begin{gathered} P^{(1)}_{2m,2m+3} = \sqrt{\frac{L}{R}}; \qquad P^{(1)}_{2m+1,2m+4} = Ra^2; \qquad P^{(1)}_{2m,2m+1} = -\frac{t}{gR}; \qquad P^{(1)}_{2m+1,2m+2} = 0 \\ R^{(1)}_{2m,2m+4} = R^{(1)}_{2m+1,2m+5} = -N\lambda; \qquad R^{(1)}_{2m,2m+2} = R^{(1)}_{2m+1,2m+3} = Nt\sqrt{RL}, \end{gathered} \tag{1.22}$$

where $a = -N\sqrt{g\lambda}$.

To calculate $S_{2N}^{(1)}(x,y)$, we use (1.20) and (1.21) in (1.7). The first four terms (modulo $O(N^{-1})$) in (1.7) give $-(2\cos 2\theta\sin(\alpha_{2N-1}))/(\pi\sin^2\theta)$, the next four terms give $(2\cos^2\theta\sin(\alpha_{2N-1}))/(\pi\sin^2\theta)$ while the last two terms give $(\sin(\alpha_{2N-1}))/\pi$, where $\alpha_j = \Delta\theta(\partial f_j(\theta))/\partial\theta)$. Combining all, we get for $x = y+\Delta y$

$$(x-y)S_{2N}^{(1)}(x,y) = \frac{\sin(\alpha_{2N-1})}{\pi} + O(N^{-1}) \qquad \Rightarrow \qquad S_{2N}^{(1)}(x,y) = \frac{\sin\left[\Delta y 2N\sqrt{g}y\sqrt{1-q^2}\right]}{\pi\Delta y} + O(N^{-1}). \quad (1.23)$$

To obtain the level-density, we take the limit:

$$\lim_{\Delta y\to 0}\frac{1}{2N}S_{2N}^{(1)}(x,y) = \frac{1}{2N}S_{2N}^{(1)}(y,y) = \frac{\sqrt{g}}{\pi}|y|\sqrt{1-q^2} + O(N^{-1}) \qquad = \frac{|y|}{\pi}\sqrt{g-\left(\frac{gy^2+t}{2}\right)^2} + O(N^{-1}), \quad (1.24)$$

while $S_{2N}^{(1)}(x,y)/S_{2N}^{(1)}(y,y)$ gives the sine-kernel:

$$\frac{S_{2N}^{(1)}(x,y)}{S_{2N}^{(1)}(y,y)} = \frac{\sin\pi r}{\pi r}, \qquad r = \Delta y S_{2N}^{(1)}(y,y). \quad (1.25)$$

2. SOP AND SYMPLECIC ENSEMBLE

For sop with quartic weight, we expand $\phi_m^{(4)}(x) = \gamma_m^{(m)}\phi_m^{(2)}(x) + \sum^{m-1}\gamma_k^{(m)}\phi_k^{(4)}(x)$. Using skew-orthonormality (1.4), we get recursion relations:

$$\phi_{2m+1}^{(4)}(x) = \gamma_{2m+1}^{(2m+1)}\phi_{2m+1}^{(2)}(x) + \gamma_{2m-1}^{(2m+1)}\phi_{2m-1}^{(4)}(x) \quad (2.1)$$

$$\phi_{2m}^{(4)}(x) = \gamma_{2m}^{(2m)}\phi_{2m}^{(2)}(x) + \gamma_{2m-2}^{(2m)}\phi_{2m-2}^{(4)}(x) + \gamma_{2m-4}^{(2m)}\phi_{2m-4}^{(4)}(x) \quad (2.2)$$

$$\psi_{2m+1}^{(4)}(x) = \gamma_{2m+1}^{(2m+1)}\left[P_{2m+1,2m+4}^{(2)}\phi_{2m+4}^{(2)}(x) + \ldots P_{2m+1,2m-2}^{(2)}\phi_{2m-2}^{(2)}(x)\right] + \gamma_{2m-1}^{(2m+1)}\psi_{2m-1}^{(4)}(x) \quad (2.3)$$

$$\psi_{2m}^{(4)}(x) = \gamma_{2m}^{(2m)}\left[P_{2m,2m+3}^{(2)}\phi_{2m+3}^{(2)}(x) + \ldots P_{2m,2m-3}^{(2)}\phi_{2m-3}^{(2)}(x)\right] + \gamma_{2m-2}^{(2m)}\psi_{2m-2}^{(4)}(x) + \gamma_{2m-4}^{(2m)}\psi_{2m-4}^{(4)}(x), \quad (2.4)$$

where $\phi_j^{(2)}(x)$ are op defined in (1.13) and (1.14).

Using skew-orthogonality relations $((\psi_{2m-2}^{(4)}(x), \phi_{2m+1}^{(4)}(x)) = 0)$, $((\psi_{2m-3}^{(4)}(x), \phi_{2m}^{(4)}(x)) = 0)$ and $((\psi_{2m-1}^{(4)}(x), \phi_{2m}^{(4)}(x)) = 0)$ in (2.1) - (2.4), we get:

$$\gamma_{2m-1}^{(2m+1)} = 2\gamma_{2m+1}^{(2m+1)}\gamma_{2m-2}^{(2m-2)}P_{2m-2,2m+1}^{(2)}; \qquad \gamma_{2m-4}^{(2m)} = -2\gamma_{2m}^{(2m)}\gamma_{2m-3}^{(2m-3)}P_{2m-3,2m}^{(2)}$$

$$\gamma_{2m-2}^{(2m)} = 2\gamma_{2m}^{(2m)}\left[\gamma_{2m-1}^{(2m-1)}P_{2m-1,2m}^{(2)} + \gamma_{2m-3}^{(2m-1)}\gamma_{2m-3}^{(2m-3)}P_{2m-3,2m}^{(2)}\right], \quad (2.5)$$

respectively. We choose $\gamma_{2m+1}^{(2m+1)} = 1/\sqrt{2}$ and $\gamma_{2m-1}^{(2m+1)} = -\sqrt{R_{2m+1}/R_{2m}}$. $\gamma_{2m-2}^{(2m-2)}$, $\gamma_{2m-2}^{(2m)}$ and $\gamma_{2m-4}^{(2m)}$ can be calculated from (1.14) and (2.5). Choice of $\gamma_{2m+1}^{(2m+1)}$ and $\gamma_{2m-1}^{(2m+1)}$ is such that we can use properties of orthogonal polynomials to solve (2.1). However, this makes the sop non-monic.

Using (1.14), (1.15) and (2.5) we solve (2.1). $\psi_{2m}^{(4)}(x)$ is obtained using skew-orthonormalization (1.4):

$$\phi_{2m+1}^{(4)}(x) = \frac{\sqrt{R_{2m+2}}}{x\sqrt{2}}\left[\phi_{2m+2}^{(2)}(x) - \phi_{2m+2}^{(2)}(0)\exp[-NV(x)]\right], \qquad \psi_{2m}^{(4)}(x) = -\frac{x\phi_{2m+2}^{(2)}(x)}{\sqrt{2R_{2m+2}}}. \quad (2.6)$$

$\phi_{2m}^{(4)}(x)$ and $\psi_{2m+1}^{(4)}(x)$ can be derived by integrating and differentiating $\psi_{2m}^{(4)}(x)$ and $\phi_{2m+1}^{(4)}(x)$ respectively.

With $m, N\to\infty$ and in the range $x_1+\delta < x < x_2-\delta$, $\delta > 0$, and neglecting the m independent term, we get

$$\phi_{2m+1}^{(4)}(x) \simeq C\sqrt{\frac{2L}{x\sin\theta}}\left[\cos\left[f_{2m+2}(\theta)\right] + O(1/N)\right], \qquad \phi_{2m}^{(4)}(x) \simeq -\frac{C}{N\lambda'}\sqrt{\frac{Rx}{2\sin^3\theta}}\left[\sin\left[f_{2m+2}(\theta)\right] + O(1/N)\right],$$

$$\psi_{2m+1}^{(4)}(x) \simeq -\left[\frac{4NC\lambda'}{\sqrt{2R}}\right]\sqrt{x\sin\theta}\left[\sin\left[f_{2m+2}(\theta)\right] + O(1/N)\right], \qquad \psi_{2m}^{(4)}(x) \simeq -C\sqrt{\frac{2x^3}{L\sin\theta}}\left[\cos\left[f_{2m+2}(\theta)\right] + O(1/N)\right], \quad (2.7)$$

where $f_m(\theta)$, $C_m \equiv C$, χ, λ and θ are defined in (1.18) and (1.20). Here, we note, that for a given x, θ varies with m. We have chosen $\theta \equiv \theta_{2m+2}$ such that

$$\phi^{(4)}_{2m+1\pm k}(x) \simeq C\sqrt{\frac{2L}{x\sin\theta}}\left[\cos\left[f_{2m+2}(\theta) \mp \frac{k\theta}{2}\right] + O(1/N)\right], \qquad m >> k, \tag{2.8}$$

and so on for $\phi^{(4)}_{2m\pm k}(x)$, $\psi^{(4)}_{2m+1\pm k}(x)$, $\psi^{(4)}_{2m\pm k}(x)$.

For the recursion coefficients (1.8), we simply read off from (2.1) - (2.4) and use (1.14). For large m we have:

$$\begin{gathered} P^{(4)}_{2m,2m+3} = -\sqrt{\frac{R}{L}}, \qquad P^{(4)}_{2m+1,2m+4} = -a^2 L, \qquad P^{(4)}_{2m,2m+1} = \frac{t}{gL}, \qquad P^{(4)}_{2m+1,2m+2} = 0, \\ R^{(4)}_{2m,2m+4} = R^{(1)}_{2m+1,2m+5} = -N\lambda, \qquad R^{(4)}_{2m,2m+2} = R^{(1)}_{2m+1,2m+3} = Nt\sqrt{RL}. \end{gathered} \tag{2.9}$$

To calculate $S^{(4)}_{2N}(x,y)$, we use (2.7), (2.8) and (2.9) in (1.7) for $\beta = 4$. The first four terms (modulo $O(N^{-1})$) in (1.7) for $\beta = 4$ give $-(\cos 2\theta \sin(\alpha_{2N+2}))/(\pi \sin^2\theta)$, the next four terms give $(\cos^2\theta \sin(\alpha_{2N+2}))/(\pi\sin^2\theta)$ while the last two terms give $(\sin(\alpha_{2N+2}))/\pi$. Combining all, we get for $x = y + \Delta y$,

$$(y-x)S^{(4)}_{2N}(x,y) = -\frac{\sin(\alpha_{2N+2})}{2\pi} + O(N^{-1}) \qquad \Rightarrow \qquad S^{(4)}_{2N}(x,y) = \frac{\sin\left[\Delta y 2N\sqrt{g}y\sqrt{1-q^2}\right]}{2\pi\Delta y} + O(N^{-1}) \tag{2.10}$$

For level-density, we take the limit

$$\lim_{\Delta y\to 0}\frac{1}{N}S^{(4)}_{2N}(x,y) = \frac{1}{N}S^{(4)}_{2N}(y,y) = \frac{\sqrt{g}}{\pi}|y|\sqrt{1-q^2} = \frac{|y|}{\pi}\sqrt{g - \left(\frac{gy^2+t}{2}\right)^2} + O(N^{-1}) \tag{2.11}$$

such that we get the "universal" sine-kernel

$$\frac{S^{(4)}_{2N}(x,y)}{S^{(4)}_{2N}(y,y)} = \frac{\sin 2\pi r}{2\pi r}, \qquad r = \Delta y S^{(4)}_{2N}(y,y). \tag{2.12}$$

3. CONCLUSION

The key achievement of this paper is the derivation of Eqs. (1.17) and (2.6). This enables us to use asymptotic results of op [1] to derive bulk asymptotics of sop with quartic weight [14, 15, 16]. Simultaneously (1.9)-(1.12) and (2.1)-(2.4) gives us the recursion coefficients $P^{(\beta)}_{j,k}$, $R^{(\beta)}_{j,k}$. These results for sop are applied in the GCD formula [8] to study quartic orthogonal and quartic symplectic ensembles of random matrices in the bulk. We note that asymptotics for sop away from the bulk can be trivially obtained from (1.17) and (2.6) since the corresponding results for op are already known.

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